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## TECHNICAL NOTE

No. 1222

### BUCKLING STRESSES OF SIMPLY SUPPORTED RECTANGULAR FLAT PLATES IN SHEAR

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FLAT PLATES IN SHEAR

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## SUMMARY

The present paper evaluates the shear buckling stresses of rectangular flat plates with simply supported edges more accurately than previous work on this problem. Both symmetric (odd number of buckles) and antisymmetric (even number of buckles) patterns were considered. A curve is presented from which the critical stresses may be obtained when the dimensions of the plate are known.

## INTRODUCTION

In reference 1 Timoshenko presents a solution for the buckling stresses of simply supported rectangular flat plates in shear. Timoshenko considered only the equations which permitted a buckle pattern symmetric about the midpoint of the plate. This limitation led to a small error in the critical stress in several cases in which the governing buckle pattern was antisymmetric instead of symmetric.

The buckling stresses have been determined more correctly than in reference 1 by considering both the symmetric and the antisymmetric buckle patterns. Through the use of the matrix iteration method described in reference 2 and by a proper choice of the terms in the series representing the deflection, more accurate results were obtained. The theoretical analysis given in the appendix is a brief summary of the derivation given in reference 1.

## SYMBOLS

T critical shear stress

 $k_s$  shear-stress coefficient which depends on length-width ratio  $\frac{a}{b}$   
of the plate

a	length of plate
b	width of plate
D	flexural stiffness of plate $\left( \frac{Et^3}{12(1 - \mu^2)} \right)$
E	Young's modulus for material
t	thickness of plate
$\mu$	Poisson's ratio for material
i, j, m, n, p, q	integers
w	deflection of plate in radial direction
x, y	axes of reference
$a_{mn}, a_{pq}, a_{ij}$	numerical coefficients

#### RESULTS AND DISCUSSION

The critical shear stress for a rectangular flat plate with simply supported edges is given by the equation

$$\tau = \frac{k_s \pi^2 D}{b^2 t}$$

Curves are presented in figure 1 giving two values of the shear-stress coefficient  $k_s$  for each value of length-width ratio from 1 to 4. These two values of  $k_s$  correspond to buckling into an odd number of buckles (symmetric buckling) and into an even number of buckles (antisymmetric buckling). Because a plate buckles into the buckle pattern which requires the least load, the solid-line curve for any given length-width ratio represents the shear-stress coefficient that corresponds to the governing buckling load. The computed values of  $k_s$  from which these curves were drawn are given in table 1. In addition, table 1 contains calculated values of deflection-function coefficients

from which buckling configurations can be drawn for each length-width ratio considered.

In figure 2 buckling configurations are shown for length-width ratios of 1.5 and 2.5 to illustrate the symmetric and antisymmetric types of buckle pattern, respectively. An equation for each of the buckling configurations was obtained by substituting into equation (3) of the appendix the values of the deflection-function coefficients given in table 1.

The values given in reference 1 for the shear-stress coefficient lie slightly above the values of the curve for symmetric buckling shown in figure 1 of the present paper (maximum deviation about 1 percent). The maximum error in the results of reference 1, which amounts to about 4 percent above the results of the present paper, occurs in the range of antisymmetric buckling.

#### CONCLUDING REMARKS

From a consideration of both symmetric and antisymmetric buckle patterns, the shear buckling stresses of rectangular flat plates with simply supported edges were more correctly evaluated than in previous work wherein only symmetric buckle patterns were considered. Through the use of the matrix iteration method and by a proper choice of the terms in the series representing the deflection, more accurate results were obtained.

Langley Memorial Aeronautical Laboratory  
National Advisory Committee for Aeronautics  
Langley Field, Va., October 25, 1946

## APPENDIX

## THEORETICAL SOLUTION

The critical stresses are determined on the basis of the principle that during buckling the elastic-strain energy stored in a structure is equal to the work done by the applied load. For a rectangular flat plate loaded in shear this equality can be written (reference 1)

$$\frac{D}{2} \int_0^b \int_0^a \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \mu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy \\ = \tau t \int_0^b \int_0^a \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dx dy \quad (1)$$

The coordinate system is shown in figure 3. Equation (1) can be rewritten in terms of the nondimensional shear-stress coefficient as follows:

$$\int_0^b \int_0^a \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \mu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right. \\ \left. - 2k_s \left( \frac{\pi}{b} \right)^2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right\} dx dy = 0 \quad (2)$$

where

$$k_s = \frac{\tau tb^2}{\pi^2 D}$$

The procedure used in solving equation (2) is to substitute for  $w$  a function of  $x$  and  $y$  that represents as closely as possible the buckling configuration and satisfies the conditions at the edges. For any buckle pattern where the value of  $w$  is 0 at all the edges, the integral of the term with the coefficient  $-2(1 - \mu)$  can be shown to vanish through the use of Green's theorem. (See reference 3.) If a series of terms with arbitrary coefficients is used to represent  $w$ , then the coefficients may be determined by the Rayleigh-Ritz method.

A general form for the deflection  $w$  satisfying the boundary conditions is

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (3)$$

If this expression for  $w$  is substituted in equation (2) the following equation is obtained:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}^2 \frac{ab}{4} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right]^2$$

$$- 2k_s \left( \frac{\pi}{b} \right)^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{mn} a_{pq} \frac{4ab}{\pi^2} \left( \frac{m\pi}{a} \right) \left( \frac{n\pi}{b} \right) \frac{pq}{(p^2 - m^2)(n^2 - q^2)} = 0 \quad (4)$$

where  $m + p$  and  $n + q$  are odd numbers.

The coefficients  $a_{mn}$  must be chosen to make the value of  $k_s$  a minimum. The minimization of  $k_s$  with respect to each  $a_{mn}$  results in the set of homogeneous linear equations represented by the following equation:

$$a_{mn} \left[ m^2 + n^2 \left( \frac{a}{b} \right)^2 \right]^2 + \frac{32k_s \left( \frac{a}{b} \right)^3}{\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{pq} \frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} = 0 \quad (5)$$

where

$$m = 1, 2, 3, \dots$$

$$n = 1, 2, 3, \dots$$

and  $m + p$  and  $n + q$  are odd numbers.

Each of the equations represented by equation (5) is associated with a specific pair of values of  $m$  and  $n$ . Since  $m + p$  and  $n + q$  are both odd,  $(m + p + n + q)$  must be even. If  $m + n$  is even,  $p + q$  must also be even; if  $m + n$  is odd,  $p + q$  must also be odd. Each of the homogeneous linear equations (5) can therefore involve only coefficients  $a_{ij}$  for which  $i + j$  is either odd or even. The set of equations (5) can therefore be divided into two independent groups which can be solved separately, one group consisting of equations in which  $i + j$  is odd and the other group consisting of equations in which  $i + j$  is even. The set of equations in which  $i + j$  is even corresponds to symmetric buckling, and the set in which  $i + j$  is odd corresponds to antisymmetric buckling. Ten equations in ten unknowns were solved for  $k_s$  for each type of buckling (symmetric and antisymmetric) by a matrix iteration method described in reference 2. The equations chosen for each particular value of  $a/b$  should contain deflection-function coefficients that give the lowest value of  $k_s$  for either type of buckling. (See reference 2.) The deflection-function coefficients for several values of  $a/b$  are given in table 1.

A representative determinant for a group of equations in which  $i + j$  is even is

	$a_{11}$	$a_{13}$	$a_{22}$	$a_{31}$	$a_{15}$	$a_{24}$	$a_{33}$	$a_{42}$	$a_{51}$	$a_{35}$
$m=1, n=1$	$M_{11}$	0	$\frac{4}{9}$	0	0	$\frac{8}{45}$	0	$\frac{8}{45}$	0	0
$m=1, n=3$	0	$M_{13}$	$-\frac{4}{5}$	0	0	$-\frac{8}{7}$	0	$-\frac{8}{25}$	0	0
$m=2, n=2$	$\frac{4}{9}$	$-\frac{4}{5}$	$M_{22}$	$-\frac{4}{5}$	$-\frac{20}{63}$	0	$-\frac{36}{25}$	0	$-\frac{20}{63}$	$\frac{4}{7}$
$m=3, n=1$	0	0	$-\frac{4}{5}$	$M_{31}$	0	$-\frac{8}{25}$	0	$\frac{8}{7}$	0	0
$m=1, n=5$	0	0	$-\frac{20}{63}$	0	$M_{15}$	$-\frac{40}{27}$	0	$-\frac{8}{63}$	0	0
$m=2, n=4$	$\frac{8}{45}$	$-\frac{8}{7}$	0	$-\frac{8}{25}$	$-\frac{40}{27}$	$M_{24}$	$-\frac{72}{35}$	0	$-\frac{8}{63}$	$\frac{8}{3}$
$m=3, n=3$	0	0	$\frac{36}{25}$	0	0	$-\frac{72}{35}$	$M_{33}$	$-\frac{72}{35}$	0	0
$m=4, n=2$	$\frac{8}{45}$	$-\frac{8}{25}$	0	$\frac{8}{7}$	$-\frac{8}{63}$	0	$-\frac{72}{35}$	$M_{42}$	$-\frac{40}{27}$	$-\frac{120}{147}$
$m=5, n=1$	0	0	$-\frac{20}{63}$	0	0	$-\frac{8}{63}$	0	$-\frac{40}{27}$	$M_{51}$	0
$m=3, n=5$	0	0	$\frac{4}{7}$	0	0	$\frac{8}{3}$	0	$-\frac{120}{147}$	0	$M_{35}$

where

$$M_{mn} = \frac{\pi^2}{32k_s \left(\frac{a}{b}\right)} \left[ m^2 + n^2 \left(\frac{a}{b}\right)^2 \right]^2$$

At a length-width ratio of 1 the lowest value of  $k_s$  that satisfies this determinant is less than the lowest value of  $k_s$  obtained from any tenth-order determinant in which  $i + j$  is odd. A representative determinant for a group of equations in which  $i + j$  is odd is

	$a_{12}$	$a_{21}$	$a_{23}$	$a_{32}$	$a_{41}$	$a_{43}$	$a_{52}$	$a_{61}$	$a_{54}$	$a_{63}$
$m=1, n=2$	$M_{12}$	$-\frac{4}{9}$	$\frac{4}{5}$	0	$-\frac{8}{45}$	$\frac{8}{25}$	0	$-\frac{4}{35}$	0	$\frac{36}{175}$
$m=2, n=1$	$\frac{4}{9}$	$M_{21}$	0	$\frac{4}{5}$	0	0	$\frac{20}{63}$	0	$\frac{8}{63}$	0
$m=2, n=3$	$\frac{4}{5}$	0	$M_{23}$	$-\frac{36}{25}$	0	0	$-\frac{4}{7}$	0	$\frac{40}{49}$	0
$m=3, n=2$	0	$\frac{4}{5}$	$-\frac{36}{25}$	$M_{32}$	$-\frac{8}{7}$	$\frac{72}{35}$	0	$-\frac{4}{9}$	0	$\frac{4}{5}$
$m=4, n=1$	$-\frac{8}{45}$	0	0	$-\frac{8}{7}$	$M_{41}$	0	$\frac{40}{27}$	0	$\frac{16}{27}$	0
$m=4, n=3$	$\frac{8}{25}$	0	0	$\frac{72}{35}$	0	$M_{43}$	$-\frac{8}{3}$	0	$\frac{80}{21}$	0
$m=5, n=2$	0	$\frac{20}{63}$	$-\frac{4}{7}$	0	$\frac{40}{27}$	$-\frac{8}{3}$	$M_{52}$	$-\frac{20}{11}$	0	$\frac{36}{11}$
$m=6, n=1$	$-\frac{4}{35}$	0	0	$-\frac{4}{9}$	0	0	$-\frac{20}{11}$	$M_{61}$	$-\frac{8}{11}$	0
$m=5, n=4$	0	$\frac{8}{63}$	$\frac{40}{49}$	0	$\frac{16}{27}$	$\frac{80}{21}$	0	$-\frac{8}{11}$	$M_{54}$	$-\frac{360}{77}$
$m=6, n=3$	$\frac{36}{175}$	0	0	$\frac{4}{5}$	0	0	$\frac{36}{11}$	0	$-\frac{360}{77}$	$M_{63}$

where

$$M_{mn} = \frac{\pi^2}{32k_s \left(\frac{a}{b}\right)^3} \left[ m^2 + n^2 \left(\frac{a}{b}\right)^2 \right]^2$$

At a length-width ratio of 2.5 the lowest value of  $k_s$  that satisfies this determinant is less than the lowest value of  $k_s$  obtained from any tenth-order determinant in which  $i + j$  is even.

#### REFERENCES

1. Timoshenko, S.: Theory of Elastic Stability. McGraw-Hill Book Co., Inc., 1936, pp. 325, 357-363.
2. Batdorf, S. B., and Stein, Manuel: Critical Combinations of Shear and Direct Stress for Simply Supported Rectangular Flat Plates. NACA TN No. 1223, 1946.
3. Southwell, R. V., and Skan, Sylvia W.: On the Stability under Shearing Forces of a Flat Elastic Strip. Proc. Roy. Soc. (London), ser. A, vol. 105, no. 733, May 1, 1924, pp. 582-607.

TABLE 1

SHEAR-STRESS COEFFICIENTS AND DEFLECTION-FUNCTION COEFFICIENTS  
FOR VARIOUS LENGTH-WIDTH RATIOS OF PLATE

[An equation for the buckling configuration for each length-width ratio can be obtained by substituting the values of the deflection-function coefficients into equation (3) of the appendix. Values of the coefficients not given in this table are assumed to be zero.]

Length-width ratio, $a/b$	Shear-stress coefficient, $k_s$		Deflection-function coefficients, $a_{ij}$ (relative magnitudes)															
			$i + j$ even															
	$i + j$	Even	Odd	$a_{11}$	$a_{13}$	$a_{22}$	$a_{31}$	$a_{15}$	$a_{24}$	$a_{33}$	$a_{42}$	$a_{51}$	$a_{35}$	$a_{44}$	$a_{53}$	$a_{62}$	$a_{71}$	$a_{91}$
1.0	9.35	11.63	1.000	-0.070	0.293	-0.071	-0.005	0.005	0.038	0.003	-0.004	0.005						
1.2	8.00	9.70	1.000	-0.053	.290	-0.088		.004	.036	.018	-0.008		0.003	0.006				
1.5	7.07	7.97	1.000	-0.042	.304	-0.147			.040	.005	-0.011		.003	.007	-0.001			
2.0	6.59	6.61	1.000	-0.032	.343	-0.325	-0.002	.005	.048	-0.042	-0.010			.004				
4.0	5.67	5.77	-0.153		.209	1.000			-0.048	-0.324	-0.300			.032	.012	-0.032	-0.010	
	i + j odd																	
	$i + j$	Even	Odd	$a_{12}$	$a_{21}$	$a_{23}$	$a_{32}$	$a_{41}$	$a_{43}$	$a_{52}$	$a_{61}$	$a_{54}$	$a_{63}$	$a_{81}$				
	2.5	6.29	6.06	0.198	1.000	-0.052	-0.316	-0.200	0.035	-0.003	-0.021	-0.002	0.008					
3.0	6.04	5.89	.150	1.000	-0.042	-0.340	-0.333	.040	.028	-0.020		.005	-0.006					

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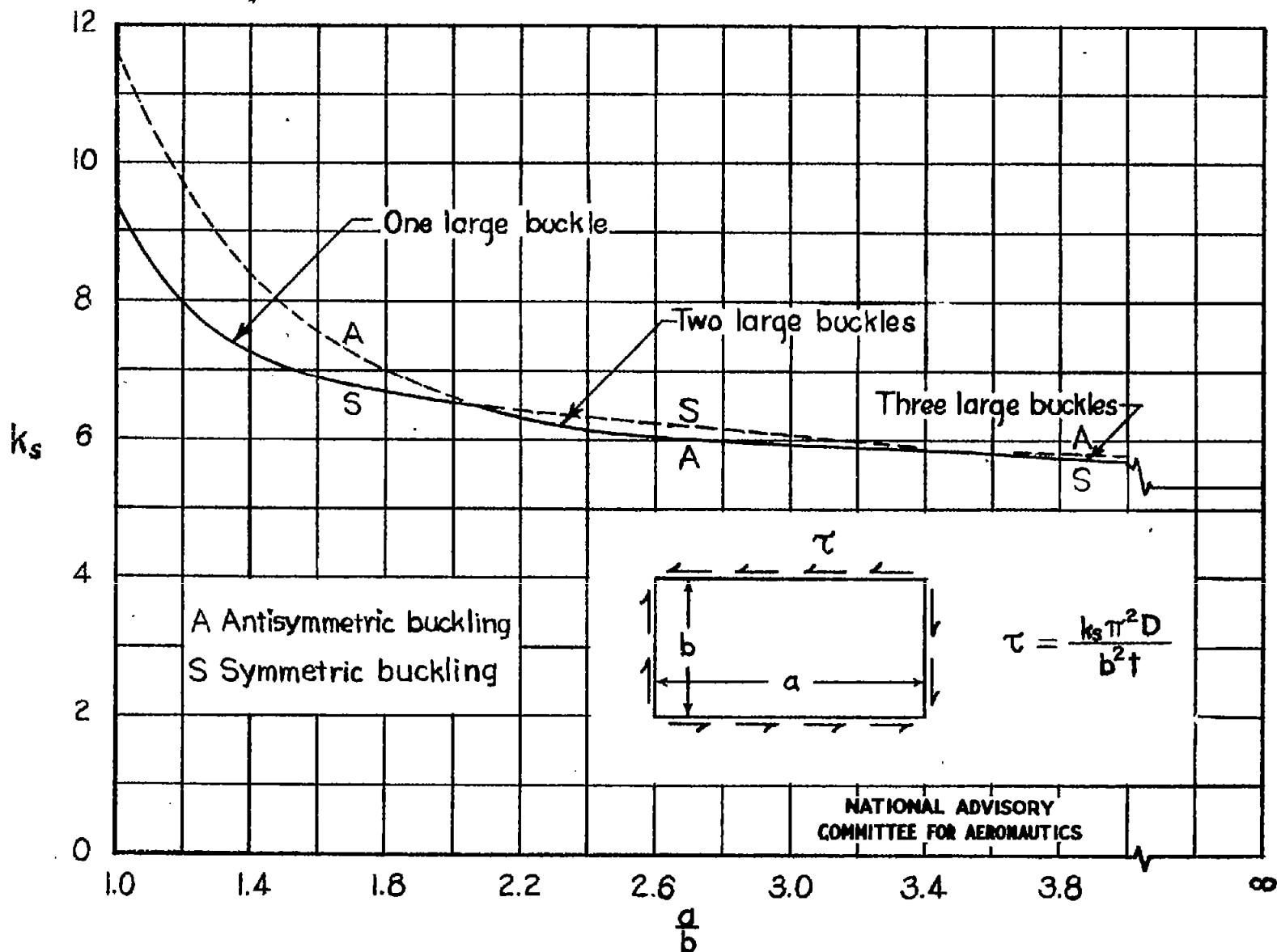
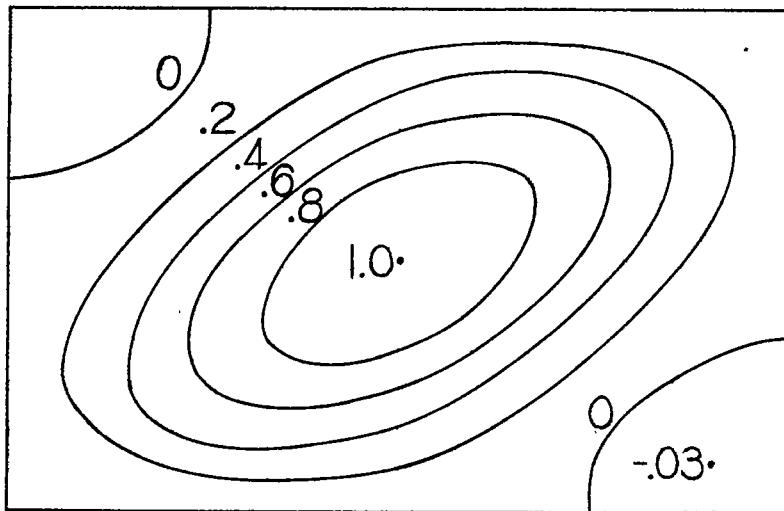
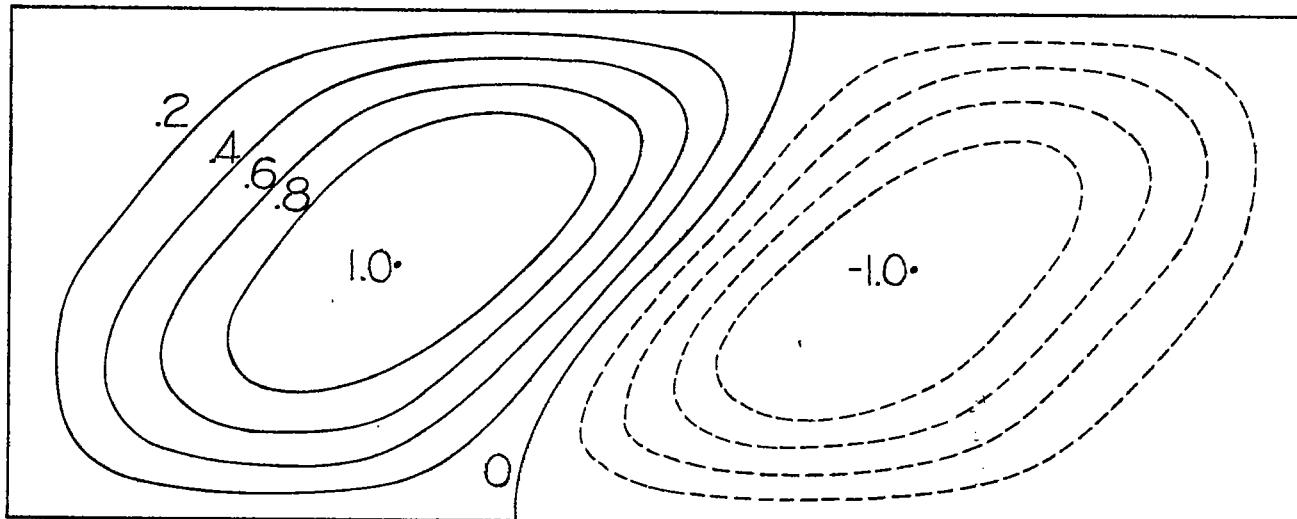


Figure 1.- Buckling stresses of a simply supported rectangular plate in shear.



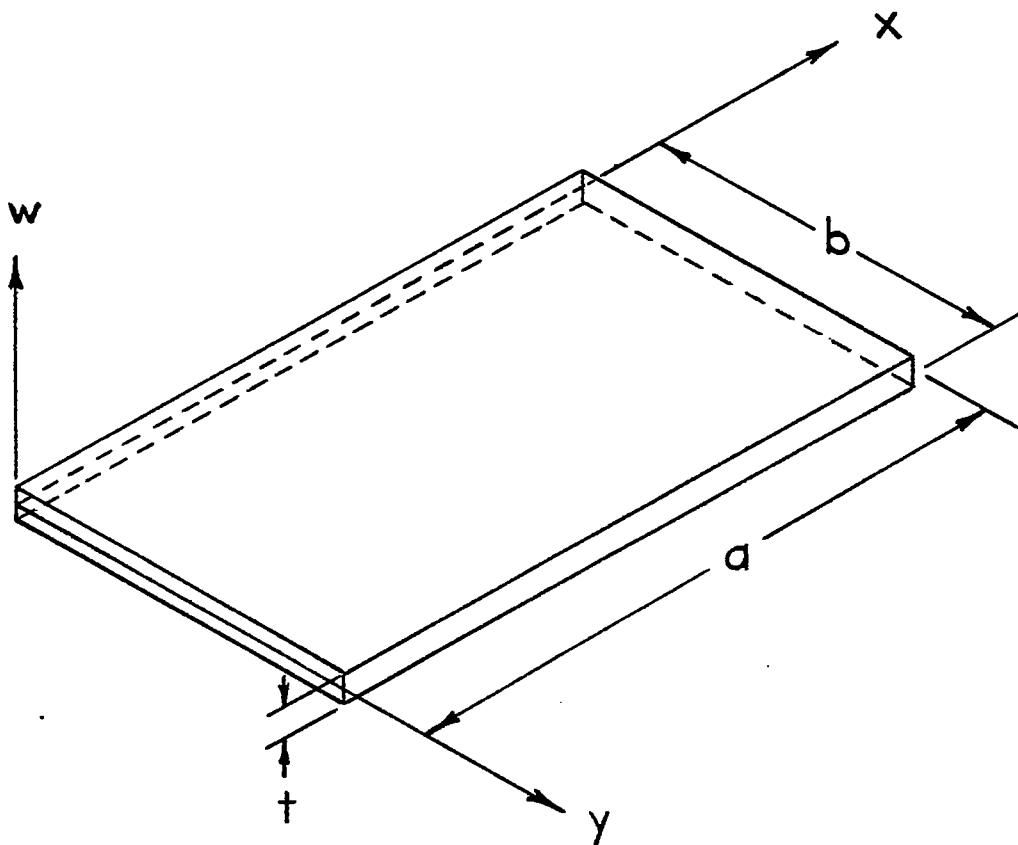
(a) Length-width ratio, 1.5; one large buckle (symmetric buckling).



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(b) Length-width ratio, 2.5; two large buckles (antisymmetric buckling).

Figure 2.- Buckling configurations of simply supported rectangular flat plates in shear.



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Figure 3.- Coordinate system for rectangular flat plate.